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Hermite Based Collocation Method for the Solution of Boundary Layer Flow of Eyring–Powell Fluid

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Abstract Eyring-Powell equations are used in computational fluid mechanics. When Mach number approaches to 0.9 or close to turbulent flow, the Newtonian equations are not accurateenough . In this research work we consider a boundary layer flow of an Eyring-Powell fluid which is non-newtonian and passing over a wedge. We consider Eyring-Powell fluid in an infinite domain and leading nonlinear differential equation are converted into the ordinary differential equations by using similarity transformations. Then we use Pseudo-Spectral method which is actually Hermite based collocation method on this ordinary differential equation and calculate its non- dimensional velocity profile. The results for different parameters is illustrated by table and figures.

Index Terms eyring powell, hermite functions, pseudo-spectral method

I. Introduction

In engineering and mathematics, a large number of problems are analyzed for infinite domain. Spectral methods such as collocation method (CM) is one of the well-known method that are used to solve such kinds of problems. Funaro, Kavian and Shen [1] used the technique of Laguerre and Hermite polynomial that are orthogonal to infinite domain to solve boundary layer (BL) problems. For finite domain another technique to solve boundary layer problems is the orthogonal polynomials. Guo proposed a Jacobi polynomial that are used to solve BVPs in which a problem is converted from infinite domain to finite domain. Chebyshev polynomial of first kind is a special case of Jacobi polynomial. When we insert these polynomials into Jacobi differential equation this gives us a recurrence relation. Jacobi poly-nomials are also orthogonal poly-nomials and they gratify recurrence relation. Boundary layer is locus of a point at which the fluid particle becomes equal to

$$u = 0.99U_\infty.$$

Also, BL is a region in flow field inside which fluid flows with a higher relative velocity. Beyond BL there will be no relative motion. Inside a BL, rate of shear strain is not equal to zero. By newton's law of viscosity,

$$\tau = \mu \frac{du}{dy}. \quad (1)$$

We studied a boundary layer flow (BLF) past a wedge with pressure gradient. These equations were first deduced by S.W Skan and V.M Falkner and then they solve it numerically for different values. But after that a more accurate solution was given by D.R Hartree. In the model of non-Newtonian fluid Eyring powell (E-P) model are useful. Boyd defined a method in which an infinite domain is replaced by $[0l]$ such that l should be large enough.This method is known as Domain Truncation method. Also, Christov and Boyd introduced a pade approximation method to solve the problems. A set of rational Legendre functions which are mutually $\perp$ in L2(0, $+\infty$) have introduced by Guo et al. On the half line the solution of Kortewegde Vries equation spectral scheme along with pade approximation was applied. Boyd et al. [2] compared with Fourier sine, Laguerre and mapped rational Chebyshev methods after applying pseudo-spectral method on semi-infinite interval. To solve nonlinear O.D.E in a semi-infinite interval rational method and CM were used by Parand and Delkhosh [3].

In past few years Eyring-Powell (E-P) fluid gained huge popularity in scientific research because of its vast uses in many fields. E-P fluid equations are used in computational fluid mechanics. When Newtonian fluid equations are not accurate enough then we use E-P fluid equations. when Mach number approaching to 0.9 or close to turbulent flow or at high speed of air Newtonian equations does not work well so here, we need equations which gives us more accurate and authentic answer, so we use E-P fluid equations. Crane introduced the method in which velocity is proportional to distance and are used to find the classical solution of boundary layer problems. Hereafter Carragher P. took the Crane problem to analyze heat transfer and derive the Nusselt number for the range of Prandtl number. A surface that have variable surface temperature linear velocity and , its stretching problem was examined by Grubka and Bobba. In the existence of free stream velocity stagnation point flow towards a stretching surface was investigated by Mahapatra et al. The slip effects of magnetohydrodynamics BLF by an exponentially SS with blowing/suction and thermal radiation was examined by Mukhopadhyay [4]. An influence of heat absorption /generation on the stagnation point flow of nanofluid over a surface with convective boundary conditions (BC) were examined by Alsaedi et al [5]. Malvandi et al. [6] has been investigated the slip effect on the time dependent stationary point flow of nano fluid over a SS.

By using the convective BC's over a moving surface, the steady flow of an E-P fluid was studied by T Hayat [7]. Since the nonlinear problems are very complex and most of them do not have analytical solution therefore the numerical and approximate methods such as R-K method, ADM,HPM and HAM and control volume based finite element methods are used. Heat transfer of E-P fluid with variable thermal conductivity and heat flux of an exponentially SS was studied by Megahed [8]. The solutions of the BLF of an E-P non-Newtonian fluid over a linearly SS by CM has been investigated by Rahimi et al. [9]. A numerical method introduced by Parand et al. [10] that is used to solve two-dimensional flow of incompressible E-P fluid of SS. Parand et. al [11] presents a paper on rational approximation of BL non-Newtonian fluid. In this work they used Bounaker CM which is also called QLM-RBC method along with quasilinearization method to find the velocity profile. Unsaturated soil water movement equation was solved with Hermite radial basis collocation method by Jiao wang et. al. [12] Nonlinear terms are present in this equation so for this problem CM cannot be used . Thats why they used Hermite radial basis CM . And find the movement of unsaturated soil water equation. R.K Nagaich and Happy Kumar [13] used Hermite CM for the solution of 2nd order parabolic PDE . This technique is a combination. In this method roots of Legendre polynomials taken as the collocation points. They used Matlab to solve the resulting set of equations. The EP equation for BLF and heat transfer around a permeable surface was solved by Mudassar et.al. [14] They used KBM to solve the equation and compare the results for different parameters.

T.Hayat [15] studied the thermal deposition and simultaneous effects of hall current of E-P fluid. thermal deposition Hall current and convection aspects were used to do mathematical modelling. Low raynold numbers were considered. They obtained pertubation solutions for the problems.

II. Mathematical Modelling

The shear stress in the sheet is

$$\tau_{ij} = \sigma \frac{du_i}{dx_j} + \frac{1}{\beta} \mathfrak{sinh}^{-1}\left(\frac{1}{c}\frac{du_i}{dx_j}\right).$$

The 2nd approximation of $\mathfrak{sinh}^{-1}$ is given as

$$\mathfrak{sinh}^{-1}\left(\frac{1}{c}\frac{du_i}{dx_j}\right) \cong \frac{1}{c}\frac{du_i}{dx_j} - \frac{1}{6}\left(\frac{1}{c}\frac{du_i}{dx_j}\right).$$

In cartesian coordinates the continuity equation is given as

$$\frac{du}{dx} + \frac{dv}{dy} = 0.$$

For our problem the mathematical modeling is that the E-P fluid model is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \left(\theta + \frac{1}{\rho\beta c}\right)\frac{\partial^2 u}{\partial y^2} - \frac{1}{2\rho\beta c^3}\left(\frac{\partial u}{\partial y}\right)\frac{\partial^2 u}{\partial y^2},$$

where $\theta = \frac{\mu}{\rho}$ is kinematic viscosity.

In this case velocity of potential flow is proportional to power distance along the wall.

$$\begin{aligned} U_s(x) &= U_1 x^m, \\ u &= -\frac{\partial \Psi}{\partial y}, \\ v &= \frac{\partial \Psi}{\partial x}, \end{aligned}$$

with boundary condition

$$\begin{aligned} \frac{\partial \Psi}{\partial x} &= 0 = \frac{\partial \Psi}{\partial y}, y = 0, \\ \frac{\partial \Psi}{\partial y} &= -U_s(x), y = \infty. \end{aligned}$$

Above equation can be solved if we introduce

$$\eta = y\sqrt{\frac{m+1}{2}}\sqrt{\frac{U_1}{\theta}}x^{\left(\frac{m-1}{2}\right)}.$$

Then solving the above eqaution it gives

$$(1+\epsilon)\,\mathfrak{f}''' - \left(\frac{m+1}{2}\right)\epsilon\delta\mathfrak{f}'''\mathfrak{f}''^2 + \mathfrak{f}\mathfrak{f}'' - \frac{2m}{m+1}\mathfrak{f}'^2 = 0,$$

where

$$\begin{aligned} \epsilon &= \frac{1}{\rho\beta c}, \\ \delta &= \frac{U_s^3}{2c^2\theta x}, \end{aligned}$$

with boundary condition

$$\begin{aligned} \mathfrak{f}(\eta) &= 0 = \mathfrak{f}'(\eta), \eta = 0, \\ \mathfrak{f}'(\eta) &= 1, \eta = \infty. \end{aligned}$$

Now our purpose is to find the dimensionless velocity $\mathfrak{f}$ and its derivatives.

III. Mathematical Formulation

In this section we will investigate that how this method works.

Hermite Functions

Here we discuss the basic characteristics of Hermite functions. we will not use Hermite polynomial because it does not behave correctly at infinty. so for this reason we use Hermite functions which can be generated from Hermite polynomials. Hermite functions are denoted by $\widetilde{\mathcal{H}}_n(x)$ and defined as

$$\widetilde{\mathcal{H}}_n(x) = \frac{1}{\sqrt{2^n n!}} e^{-\frac{x^2}{2}} \mathcal{H}_n(x), n \geq 0, x \in \mathbb{R},$$

where $\mathcal{H}_n(x)$ is Hermite polynomial.

These Hermite functions are normalized by

$$\int_{-\infty}^{+\infty} \widetilde{\mathcal{H}}_n(x) \widetilde{\mathcal{H}}_m(x)\, dx = \sqrt{\pi}\delta_{mn},$$

where δ_{mn} is kronecker delta function. Hermite functions also satisfy decay property.

$$\left|\widetilde{\mathcal{H}}_n(x)\right| \quad \longrightarrow \quad 0 \; as \; |x| \longrightarrow \infty.$$

The asymptotic formula for Hermite functions with large n is

$$\widetilde{\mathcal{H}}_n(x) \sim n^{-\frac{1}{4}} \cos\left(\sqrt{2n+1}x - n\frac{\pi}{2}\right)$$

The recurrence relation for Hermite functions is

$$\widetilde{\mathcal{H}}_{n+1}(x) = x\sqrt{\frac{2}{n+1}}\widetilde{\mathcal{H}}_n(x) - \sqrt{\frac{n}{n+1}}\widetilde{\mathcal{H}}_{n-1}(x), n \geq 1.$$

From these equation and relation we can check the orthogonality which gives results as

$$\int_{-\infty}^{+\infty} \widetilde{\mathcal{H}}'_n(x)\widetilde{\mathcal{H}}'_m(x)dx = \left\{ \begin{array}{c} -\frac{\sqrt{n\pi(n-1)}}{2}, m = n-2 \\ \left(n+\frac{1}{2}\right)\sqrt{\pi}, m = n \\ -\frac{\sqrt{\pi(n+1)(n+2}}{2}, m = n+2 \\ 0, otherwise \end{array} \right\}.$$

Now let us define

$$\widetilde{Q}_N : \left\{ v : v = e^{-\frac{x^2}{2}}\omega, \forall\omega \in Q_N \right\},$$

where Q_N is the set of all Hermite polynomials with degree atmost N.

Next we will establish the Guass quadrature which is linked with Hermite functions.

Let $\{x_i\}_{i=0}^{N}$ be the Hermite Guass nodes so we define the weights

$$\widetilde{\omega}_i = \frac{\sqrt{\pi}}{(N+1)\widetilde{\mathcal{H}}_N^2(x_i)}, 0 \leq i \leq N.$$

Then we have

$$\int_{-\infty}^{+\infty} q(x)dx = \sum_{i=0}^{N} q(x_i)\widetilde{\omega}_i, \forall q \in \widetilde{Q}_{2N+1}.$$

Since we discussed earlier that the Eyring Powell (E-P) problem is defined in the interval of $(0, +\infty)$ but the Hermite functions are lies between $(-\infty, +\infty)$ so to handle this dificulty we use a transformation which changes the interval of Hermite function from $(0, +\infty)$ to $(-\infty, +\infty)$ and the new Hermite functions are called transformed Hermite functions.

$$\omega = \varphi(\mathfrak{z}) = \frac{1}{k}\ln(\mathfrak{z}),$$

where k is constant.

The inverse map of $\omega = \varphi(\mathfrak{z})$ is

$$\mathfrak{z} = \varphi^{-1}(\omega) = e^{k\omega}.$$

Thus we define the inverse images of space nodes $\{x_i\}_{i=-\infty}^{i=+\infty}$ as

$$\xi = \left\{\varphi^{-1}(x) : -\infty \leq x \leq +\infty\right\} = (0, +\infty),$$

and

$$\widetilde{x}_i = \varphi^{-1}(x_i) = e^{kx_i}, i = 0, 1, 2, ..., n.$$

By using this relation we will find the Hermite Guass roots.

When we put the highest transformed Hermite function equal to zero we get the roots from where we can find the collocation points.

$$\widehat{\mathcal{H}}_{N+1}(x) = 0.$$

Now we can use above equation to find the Hermite Guass roots. to obtain the better results we convert the roots from $(0, +\infty)$ interval to $[0, 12]$.

So for our E-P problem roots can be calculated as

$$\widetilde{\eta}_i = 12\frac{\widetilde{x}_i}{\widetilde{x}_n}, i = 0, 1, 2, ..., n,$$

where $\widetilde{x}_n$ is the largest root.

IV. Application of the Method

Since our equation is

$$(1+\epsilon)\left(\frac{d^3}{d\eta^3}\mathfrak{f}(\eta)\right) - \epsilon\delta\left(\frac{m+1}{2}\right)\left(\frac{d^2}{d\eta^2}\mathfrak{f}(\eta)\right)^2\left(\frac{d^3}{d\eta^3}\mathfrak{f}(\eta)\right)$$
$$-\frac{2m}{m+1}\left(\frac{d}{d\eta}\mathfrak{f}(\eta)\right)^2 + \mathfrak{f}(\eta)\left(\frac{d^2}{d\eta^2}\mathfrak{f}(\eta)\right) = 0,$$

we take

$$\begin{aligned} m &= 2, \\ \epsilon &= 0.3, \\ \delta &= 0.2. \end{aligned}$$

Now we solve this equation by collocation method. First we calculate the following function.

$$y(\eta) = \sum_{i=1}^{n+1} b_i \widehat{\mathcal{H}}_i(\eta).$$

Here b_i are the unknown coeffcients which are to be find and n is the degree of Hermite functions. Now substituting the values of $\widehat{\mathcal{H}}_i(\eta)$

In next step we satisfy the boundary condition as

$$\mathfrak{f}(\eta) = \eta + \eta^2 y(\eta).$$

This equation surely satisfies our BC. so we substitute it to our orignal equation by taking derivatives. and also substitute the values of $\epsilon = 0.3$,$\delta = 0.2$,$m = 2$,$k = 0.1$

Then we substitute our roots or collocation points in above equation one by one and we get a system of simultaneous equations.

After that we solve this non-linear system of equations to get our coeffcients b_i.

This calculation is done on the software and we get the coeffcients

$$\begin{aligned} b_1 &= -3.055533935, \\ b_2 &= -1.070783759, \\ b_3 &= .5693198643, \\ b_4 &= 2.024371832, \\ b_5 &= 2.655870499, \\ b_6 &= 4.263417093, \\ b_7 &= 5.339883004. \end{aligned}$$

Substituting these values in above equation and simplifying we get the solution which is

$$\begin{aligned} \mathfrak{f}(\eta) = \; & 1.210432990 \times 10^{-11} \ln(\eta) exp(-.500000\eta^2) \\ & +7.992141131 \times 10^{-14} exp(-.500000\eta^2) \\ & -1.615271358 \times 10^{-11} exp(-.500000\eta^2) ln(\eta)^2 \\ & -3.699583112 \times 10^{-13} ln(\eta)^3 exp(-.500000\eta^2) \\ & +1.117949736 \times 10^{-11} ln(\eta)^4 exp(-.500000\eta^2) \\ & -7.399981327 \times 10^{-12} ln(\eta)^5 exp(-.500000\eta^2) \\ & +1.97030385 \times 10^{-12} ln(\eta)^6 exp(-.500000\eta^2) \\ & +1.944487611 \times 10^{-13} ln(\eta)^6 exp(-.500000\eta^2). \end{aligned}$$

V. Results and Discussion

In this research work the Hermite CM is applied on Eyring-Powell (E-P) boundary layer equations past a wedge. Figures 1 and 2 show the results for $\mathfrak{f}'(\eta)$ and $\mathfrak{f}''(\eta)$ which are get from Hermite CM for various value of η.The result for dimensional-less velocity field $\mathfrak{f}'(\eta)$on different values of η have shown in Table 1.The effect of ϵ on $\mathfrak{f}'(\eta)$ and $\mathfrak{f}''(\eta)$ have shown in figure 1 and 2 respectively.It is observed that when we increase the value of ϵ its increase the dimensional-less velocity field $\mathfrak{f}'(\eta)$as shown in figure 1.The behavior of $\mathfrak{f}'(\eta)$ and $\mathfrak{f}''(\eta)$ on different values of δ are shown in figure 3 & 4 respectively.

The effects of E-P fluid parameters of ϵ and δ is shown in Figure 3 and 4. These graphs shows the variation in velocity field and its derivative for ϵ =0.1 and distinct values of δ.

VI. Conclusion

In this work we successfully apply Pseudo-Spectral method to solve approximate EPBL equation past a wedge.The effects of Non-newtonian EPF parameter and on velocity and stress profile observed when fluid past a wedge. By increasing the values of ϵ. $\mathfrak{f}'$ & $\mathfrak{f}''$ also increases and by increasing the values δ the value of $\mathfrak{f}'$and $\mathfrak{f}''$ decreases. Its noted that this technique can be used to solve other non-linear problems.

η	P-S
0	1
0.2	0.828358
0.4	07133983
0.6	0.581209
0.8	0.492247
1.0	0.404135
1.2	0.323876
1.4	0.292011
1.6	0.221345
1.8	0.214893
2.0	0.189838
2.2	0.132039
2.4	0.129840
2.6	0.112134
2.8	0.098434
3.0	0.070896
3.2	0.058324
3.4	0.050340
3.6	0.041293
3.8	0.031673
4.0	0.026989
4.2	0.025098
4.4	0.019821
4.6	0.018689
4.8	0.013412
5.0	0.011453

Table 1: The results for distinct values of η for $\mathfrak{f}'(\eta)$

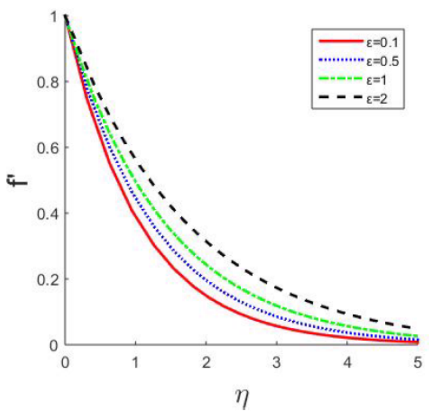

Figure 1: Comparison of answers of velocity fields $\mathfrak{f}'(\eta)$ on distinct values of ϵ

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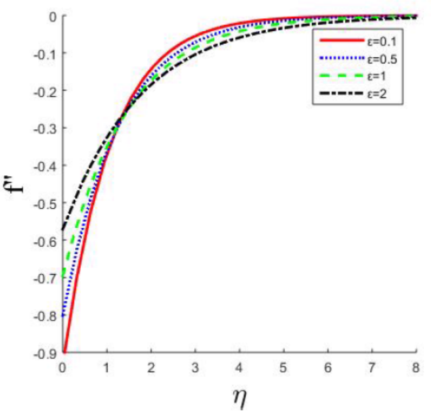

Figure 2: Comparison of answers of $\mathfrak{f}''(\eta)$ on distinct values of ϵ

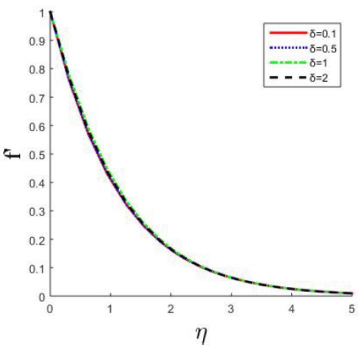

Figure 3: Comparison of answers of velocity fields $\mathfrak{f}'(\eta)$ on distinct values of δ

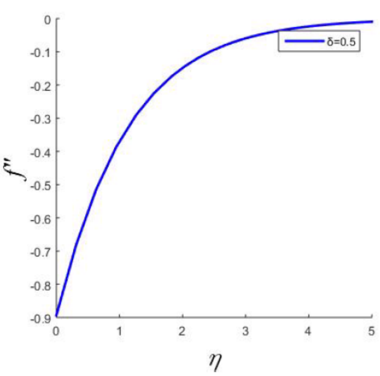

Figure 4: Behavior of $\mathfrak{f}''(\eta)$ on distinct values of δ

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