Publication Date: 31 July 2024
Archs Sci. (2024) Volume 74, Issue 4 Pages 16-22, Paper ID 2024403.
https://doi.org/10.62227/as/74403

Hermite-Hadamard Inequalities for (h, m)-convex Modified Functions of Second Type via Weighted Integral

Paulo M. Guzmán[1,2], **Juan E. Nápoles V.**[2,3], **Murat Cancan**[4] **and Saeid Jafari**[1,*]
[1]Facultad de Ciencias Agrarias, Universidad Nacional del Nordeste, Sargento Cabral 2131, Corrientes 3400, Argentina.
[2]Facultad de Ciencias Exactas y Naturales y Agrimensura, Universidad Nacional del Nordeste, Av. Libertad 5450, Corrientes 3400, Argentina.
[3]Facultad Regional Resistencia, Universidad Tecnológica Nacional, French 414, Resistencia 3506, Argentina.
[4]Faculty of Education,Van Yuzuncu Yil University, Zeve Campus, Tusba, 65080, Van, Turkey.
[5]Mathematical and Physical Science Foundation, 4200 Slagelse, Denmark.

Corresponding authors: (jafaripersia@gmail.com).

Abstract In this work we obtain integral inequalities of the Hermite-Hadamard type, using weighted integral, via (h, m)-convex modified functions of second type. In the work we show that several results known from the literature can be derived from ours as particular cases.

Index Terms Integral inequalities, (h, m)-convex modified functions of second type, weighted integral

I. Introduction

The concept of convex function is a notion that is among the most well-known functional classes in Mathematical Sciences today. This is due to their properties, geometric characteristics, interpretation and the variety of areas where they are used and applied.

The essence of its definition is the relationship and comparison between two midpoints, that of the interval $[a, b]$ and that of the interval $[f(a), f(b)]$. Below we remember this definition.

Definition 1. A function $f : I \to \mathbb{R}$ is said to be convex, if the inequality

$$\psi(\tau\xi + (1-\tau)\varsigma) \leq \tau\psi(\xi) + (1-\tau)\psi(\varsigma) \tag{1}$$

holds for all $\xi, \varsigma \in I$ and $\tau \in [0, 1]$.

The interest in this notion has resulted in a large number of extensions and ramifications that make it practically impossible to follow. Interested readers can consult [1] to get a more precise idea of this development.

One of the most important inequalities, for convex functions, is the very well known Hermite–Hadamard inequality, that is the inequality

$$\psi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \psi(x)dx \leq \frac{\psi(a)+\psi(b)}{2} \tag{2}$$

holds for any function ψ convex on the interval $[a, b]$. This inequality was published by Hermite ([2]) in 1883 and, independently, by Hadamard in 1893 ([3]), it gives an estimation of the mean value of a convex function. Various extensions and generalizations can be consulted in [4]–[19] and references therein.

In [7], the following definitions were presented:

Definition 2. Let $h : [0, 1] \to \mathbb{R}$ be a nonnegative function, $h \neq 0$ and $\psi : I = [0, +\infty) \to [0, +\infty)$. If inequality

$$\psi(\tau\xi + m(1-\tau)\varsigma) \leq h^s(\tau)\psi(\xi) + m(1-h^s(\tau))\psi(\varsigma) \tag{3}$$

holds for all $\xi, \varsigma \in I$ and $\tau \in [0, 1]$, where $m \in [0, 1]$, $s \in [-1, 1]$. Then a function ψ is called a (h, m)-convex modified of the first type on I.

Definition 3. Let $h : [0, 1] \to \mathbb{R}$ nonnegative functions, $h \neq 0$ and $\psi : I = [0, +\infty) \to [0, +\infty)$. If inequality

$$\psi(\tau\xi + m(1-\tau)\varsigma) \leq h^s(\tau)\psi(\xi) + m(1-h(\tau))^s\psi(\varsigma) \tag{4}$$

holds for all $\xi, \varsigma \in I$ and $\tau \in [0, 1]$, where $m \in [0, 1]$, $s \in [-1, 1]$. Then a function ψ is called a (h, m)-convex modified of the second type on I.

Remark 1. From Definitions 2 and 3, we can define $N^s_{h,m}[a, b]$, where $a, b \in [0, +\infty)$, as the set of functions (h, m)-convex modified, for which $\psi(a) \geq 0$, characterized by the triple $(h(\tau), m, s)$. Note that if:

1) $(h(\tau), 0, 0)$ we have the increasing functions ([20]).
2) $(\tau, 0, s)$ we have the $s-$starshaped functions ([20]).
3) $(\tau, 0, 1)$ we have the starshaped functions ([20]).
4) $(\tau, 1, 1)$ then ψ is a convex function on $[0, +\infty)$ ([20]).
5) $(1, 1, s)$ then ψ is a P-convex function on $[0, +\infty)$ ([21]).
6) $(\tau, m, 1)$ then ψ is a $m-$convex function on $[0, +\infty)$ ([22]).

7) $(\tau, 1, s)$ $s \in (0, 1]$ then ψ is a $s-$convex function on $[0, +\infty)$ ([23], [24]).
8) $(\tau, 1, s)$ $s \in [-1, 1]$ then ψ is a $s-$convex extended function on $[0, +\infty)$.
9) (τ, m, s) $s \in (0, 1]$ then ψ is a (s, m)-convex extended function on $[0, +\infty)$ ([25]).
10) $(\tau^a, 1, s)$ with $a \in (0, 1]$, then ψ is a (a, s)-convex function on $[0, +\infty)$ ([26]).
11) $(\tau^a, m, 1)$ with $a \in (0, 1]$, then ψ is a (a, m)-convex function on $[0, +\infty)$ ([27]).
12) (τ^a, m, s) with $a \in (0, 1]$, then ψ is a $s-(a, m)$-convex function on $[0, +\infty)$ ([28]).
13) $(h(\tau), m, 1)$ then we have a variant of the $(h, m)-$convex function on $[0, +\infty)$ ([29]).

Next we present the weighted integral operators, which will be the basis of our work ([7], [30], [31]).

Definition 4. Let $\phi \in L([a, b])$ and let w be a continuous and positive function, $w : [0, 1] \to [0, +\infty)$, with second order derivatives integrable on I. Then the weighted fractional integrals are defined by (right and left respectively):

$$J^w_{a+}\phi(r) = \int_a^r w''\left(\frac{r-\sigma}{b-a}\right)\phi(\sigma)d\sigma, \quad r > a$$

and

$$J^w_{b-}\phi(r) = \int_r^b w''\left(\frac{\sigma-r}{b-a}\right)\phi(\sigma)d\sigma, \quad r < b.$$

Remark 2. To have a clearer idea of the amplitude of the Definition 4, let's consider some particular cases of the kernel w'':

1) Putting $w''(t) \equiv 1$, we obtain the classical Riemann integral.
2) If $w''(t) = \frac{t^{(\alpha-1)}}{\Gamma(\alpha)}$, then we obtain the Riemann-Liouville fractional integral right, and left can be obtained similarly.
3) With convenient kernel choices w'' we can get the k-Riemann-Liouville fractional integral right and left of ([32]), the right-sided fractional integrals of a function ψ with respect to another function h on $[a, b]$, the right and left integral operator of [33], the right and left sided generalized fractional integral operators of [34] and the integral operators of [35] and [36], can also be obtained from above Definition by imposing similar conditions to w''.

Of course there are other known integral operators, fractional or not, that can be obtained as particular cases of the previous one, but we leave it to interested readers (see [37], [38]).

In this work, we present new results related to the Hermite-Hadamard Inequality, for (h, m)-convex functions modified of second type, in the framework of Weighted Integral of Definition 4. We will see that several well-known publications in the literature are particular cases of those obtained by us.

II. Main Results

As a basis for proving our main results, we need the following equality.

Lemma 1. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$. If $f'' \in L[a, b]$ and w has second derivative integrable on I, then the followng equality holds:

$$\begin{aligned}
&\frac{(b-a)^2}{4}\left[w(0)\left(f'(b)-f'(a)\right)\right] \\
&+\left[w'(1)f\left(\frac{a+b}{2}\right)+w'(0)\frac{(f(b)+f(a))}{2}\right] \\
&-\frac{1}{b-a}\left(J^w_{a+}f\left(\frac{a+b}{2}\right)+J^w_{b-}f\left(\frac{a+b}{2}\right)\right) \\
&=\frac{(b-a)^2}{8}\left[\int_0^1 w(t)f''\left(\frac{a+b}{2}t+(1-t)a\right)dt\right. \\
&\left.+\int_0^1 w(1-t)f''\left(bt+(1-t)\frac{a+b}{2}\right)dt\right]. \qquad (5)
\end{aligned}$$

Proof. Let's call $I = I_1 + I_2$, so we have

$$\begin{aligned}
I_1 &= \int_0^1 w(t)f''\left(\frac{a+b}{2}t+(1-t)a\right)dt \\
&=\frac{2}{b-a}w(t)f'\left(\frac{a+b}{2}t+(1-t)a\right)|_0^1 \\
&\quad-\frac{2}{b-a}\int_0^1 w'(t)f'\left(\frac{a+b}{2}t+(1-t)a\right)dt \\
&=\frac{2}{b-a}\left[w(1)f'\left(\frac{a+b}{2}\right)-w(0)f'(a)\right] \\
&\quad-\frac{2}{b-a}\left[\frac{2}{b-a}w'(t)f\left(\frac{a+b}{2}t+(1-t)a\right)|_0^1\right. \\
&\quad\left.-\frac{2}{b-a}\int_0^1 w''(t)f\left(\frac{a+b}{2}t+(1-t)a\right)\right] \\
&=\frac{2}{b-a}\left[w(1)f'\left(\frac{a+b}{2}\right)-w(0)f'(a)\right] \\
&\quad-\left(\frac{2}{b-a}\right)^2\left[w'(1)f\left(\frac{a+b}{2}\right)-w'(0)f(a)\right. \\
&\quad\left.-\frac{2}{b-a}\int_a^{\frac{a+b}{2}} w''\left[\frac{u-a}{\frac{a+b}{2}-a}\right]f(u)du\right].
\end{aligned}$$

Analogously

$$\begin{aligned}
I_2 &= \int_0^1 w(1-t)f''\left(bt+(1-t)\frac{a+b}{2}\right)dt \\
&=\frac{2}{b-a}w(1-t)f'\left(bt+(1-t)\frac{a+b}{2}\right)|_0^1 \\
&\quad+\frac{2}{b-a}\int_0^1 w'(1-t)f'\left(bt+(1-t)\frac{a+b}{2}\right)dt
\end{aligned}$$

ARCHIVES DES SCIENCES A Multidisciplinary Journal

$$
\begin{aligned}
=&\frac{2}{b-a}\left[w(0)f'(b)-w(1)f'\left(\frac{a+b}{2}\right)\right]\\
&+\frac{2}{b-a}\left[\frac{2}{b-a}w'(1-t)f'\left(bt+(1-t)\frac{a+b}{2}\right)|_0^1\right.\\
&\left.+\frac{2}{b-a}\int_0^1 w''(1-t)f\left(bt+(1-t)\frac{a+b}{2}\right)dt\right]\\
=&\frac{2}{b-a}\left[w(0)f'(b)-w(1)f'\left(\frac{a+b}{2}\right)\right]\\
&+\left(\frac{2}{b-a}\right)^2\left[w'(0)f(b)-w'(1)f\left(\frac{a+b}{2}\right)\right.\\
&\left.+\frac{2}{b-a}\int_0^1 w''(1-t)f\left(bt+(1-t)\frac{a+b}{2}\right)dt\right]\\
=&\frac{2}{b-a}\left[w(0)f'(b)-w(1)f'\left(\frac{a+b}{2}\right)\right]\\
&+\left(\frac{2}{b-a}\right)^2\left[w'(0)f(b)-w'(1)f\left(\frac{a+b}{2}\right)\right.\\
&\left.+\frac{2}{b-a}\int_{\frac{a+b}{2}}^b w''\left[\frac{b-u}{b-\frac{a+b}{2}}\right]f(u)du\right].
\end{aligned}
$$

Adding the results obtained for I_1 and I_2 we have

$$
\begin{aligned}
I=&\frac{2}{b-a}\left[w(0)\left(f'(b)-f'(a)\right)\right]\\
&+\left(\frac{2}{b-a}\right)^2\left\{\left[2w'(1)f\left(\frac{a+b}{2}\right)\right.\right.\\
&\left.+w'(0)\left(f(b)-f(a)\right)\right]\\
&\left.+\frac{2}{b-a}\left(J^w_{a+}f\left(\frac{a+b}{2}\right)+J^w_{b-}f\left(\frac{a+b}{2}\right)\right)\right\}.
\end{aligned}
$$

Which is the desired result. □

From Lemma 1 we have the following result (see Lemma 1 of [39]).

Remark 3. Putting $w(t)=z^2$ in (5), we obtain

Corollary 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$. If $f'' \in L[a, b]$, then the following equality holds:*

$$
\begin{aligned}
&\frac{1}{b-a}\int_a^b f(u)du-f\left(\frac{a+b}{2}\right)\\
&=\int_0^1 t^2 f''\left(\frac{a+b}{2}t+(1-t)a\right)dt\\
&\quad+\int_0^1 (1-t)^2 f''\left(bt+(1-t)\frac{a+b}{2}\right)dt.
\end{aligned}
$$

Proof. From I_1 we have

$$
\begin{aligned}
&\int_0^1 t^2 f''\left(\frac{a+b}{2}t+(1-t)a\right)dt\\
&=\frac{2}{b-a}f'\left(\frac{a+b}{2}\right)-2\left(\frac{2}{b-a}\right)^2 f\left(\frac{a+b}{2}\right)\\
&\quad+\frac{16}{(b-a)^3}\int_a^{\frac{a+b}{2}} f(u)du,
\end{aligned}
$$

and from I_2:

$$
\begin{aligned}
&\int_0^1 (1-t)^2 f''\left(bt+(1-t)\frac{a+b}{2}\right)dt\\
&=-\frac{2}{b-a}f'\left(\frac{a+b}{2}\right)-2\left(\frac{2}{b-a}\right)^2 f\left(\frac{a+b}{2}\right)\\
&\quad+\frac{16}{(b-a)^3}\int_a^{\frac{a+b}{2}} f(u)du.
\end{aligned}
$$

Adding these two results we obtain the searched result. □

A new result for fractional Riemann-Liouville integrals can be obtained from Lemma 1 as follows.

Remark 4. Putting $w(t) = \frac{z^{\alpha+1}}{\Gamma(\alpha)}$ in (5) we obtain the following result:

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$. If $f'' \in L[a, b]$, then the followng equality holds:

$$
\begin{aligned}
&-\frac{1+\alpha}{\Gamma(\alpha)}f\left(\frac{a+b}{2}\right)+\frac{\alpha(1+\alpha)2^{\alpha-1}}{(b-a)^\alpha}\left(J^\alpha_{a+}f\left(\frac{a+b}{2}\right)\right.\\
&\left.+J^\alpha_{b-}f\left(\frac{a+b}{2}\right)\right)\\
&=\frac{(b-a)^2}{8\Gamma(\alpha)}\left[\int_0^1 t^{\alpha+1}f''\left(\frac{a+b}{2}t+(1-t)a\right)dt\right.\\
&\left.+\int_0^1 (1-t)^{\alpha+1}f''\left(bt+(1-t)\frac{a+b}{2}\right)dt\right].
\end{aligned}
$$

Theorem 1. *Let $f : I \subset [0, d) \to \mathbb{R}$ b a differentiable mapping on I°, such that $f'' \in L[a, b]$, where $a, b \in I$ with $0 \le a < b < \infty$ and w has second derivative integrable on I. If $|f''|$ is (h, m)-convex on $[a, b]$, for some fixed $m \in (0, 1]$, $t \in [0, 1]$, then the following inequality holds:*

$$
\begin{aligned}
|L|\le&\frac{(b-a)^2}{8}\left\{\int_0^1\left[w(t)\left|f''\left(\frac{a+b}{2}\right)\right|\right.\right.\\
&\left.+w(1-t)\left|f''(b)\right|\right]h^s(t)dt\\
&+m\int_0^1\left[w(t)\left|f''\left(\frac{a}{m}\right)\right|+w(1-t)\left|f''\left(\frac{a+b}{2m}\right)\right|\right]\\
&\left.\cdot(1-h(t))^s dt\right\} \qquad (6)
\end{aligned}
$$

with

$$
\begin{aligned}
L=&\frac{b-a}{4}\left[w(0)\left(f'(b)-f'(a)\right)\right]+\left[w'(1)f\left(\frac{a+b}{2}\right)\right.\\
&\left.+w'(0)\frac{f(a)+f(b)}{2}\right]\\
&-\frac{1}{b-a}\left(J^w_{a+}f\left(\frac{a+b}{2}\right)+J^w_{b-}f\left(\frac{a+b}{2}\right)\right).
\end{aligned}
$$

Proof. From (5) we have

$$
|L|=\frac{(b-a)^2}{8}\left|I_1+I_2\right|\le\frac{(b-a)^2}{8}\left(|I_1|+|I_2|\right), \qquad (7)
$$

with

$$I_1=\int_0^1 w(t)f''\left(\frac{a+b}{2}t+(1-t)a\right)dt,$$
$$I_2=\int_0^1 w(1-t)f''\left(tb+(1-t)\frac{a+b}{2}\right)dt.$$

So using the (h,m)-convexity of $|f''|$ we have

$$\begin{aligned}|I_1|&=\left|\int_0^1 w(t)f''\left(\frac{a+b}{2}t+(1-t)a\right)dt\right|\\&\le\int_0^1 w(t)\left|f''\left(\frac{a+b}{2}t+(1-t)a\right)\right|dt\\&\le\int_0^1 w(t)\Bigg(\left|f''\left(\frac{a+b}{2}\right)\right|h^s(t)\\&\quad+m\left|f''\left(\frac{a}{m}\right)\right|(1-h(t))^s\Bigg)dt.\end{aligned}\tag{8}$$

Analogously

$$\begin{aligned}|I_2|\le&\int_0^1 w(1-t)\Bigg(|f''(b)|h^s(t)\\&+m\left|f''\left(\frac{a+b}{2m}\right)\right|(1-h(t))^s\Bigg)dt.\end{aligned}\tag{9}$$

Substituting (8) and (9) into (7) we obtain the desired result. □

Remark 5. With $s=1$, and $w(z)=z^2$ we have the Theorem 2.2 of [40], obviously under the above assumptions the Remark 2.3 of above paper is also valid.

Remark 6. With $w(z)=\frac{z^{\alpha+1}}{\Gamma(\alpha)}$ we have the following result for Riemann-Liouville integrals:

Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° where $a,b\in I$ with $a<b$. If $f''\in L[a,b]$, then the followng equality holds:

$$\begin{aligned}&\Bigg|\frac{\alpha(1+\alpha)2^{\alpha-1}}{(b-a)^\alpha}\left(J^\alpha_{a+}f\left(\frac{a+b}{2}\right)+J^\alpha_{b-}f\left(\frac{a+b}{2}\right)\right)\\&\quad-\frac{1+\alpha}{\Gamma(\alpha)}f\left(\frac{a+b}{2}\right)\Bigg|\\&\le\frac{(b-a)^2}{8\Gamma(\alpha)}\Bigg[\int_0^1\left[t^{\alpha+1}\left|f''\left(\frac{a+b}{2}\right)\right|+(1-t)^{\alpha+1}|f''(b)|\right]\\&\quad\cdot h^s(t)dt\\&\quad+m\int_0^1\left[t^{\alpha+1}\left|f''\left(\frac{a}{m}\right)\right|+(1-t)^{\alpha+1}\left|f''\left(\frac{a+b}{2}\right)\right|\right]\\&\quad\cdot(1-h(t))^s dt\Bigg].\end{aligned}$$

The following result establishes a new refinement to the previous Theorem.

Theorem 2. *Let $f:I\subset[0,d)\to\mathbb{R}$ b a differentiable mapping on I°, such that $f''\in L[a,b]$, where $a,\ b\in I$ with $0\le a<b<\infty$ and w has second derivative integrable on I. If $|f''|^q$ is (h,m)-convex on $[a,b]$, for some fixed $m\in(0,1]$, $t\in[0,1]$ and $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:*

$$\begin{aligned}|L|\le&\frac{(b-a)^2}{8}\Bigg\{\left(\int_0^1 w^p(t)dt\right)^{\frac{1}{p}}\Bigg(\left|f''\left(\frac{a+b}{2}\right)\right|^q\\&\times\int_0^1 h^s(t)dt+m\left|f''\left(\frac{a}{m}\right)\right|^q\int_0^1(1-h(t))^s dt\Bigg)^{\frac{1}{q}}\\&+\left(\int_0^1 w^p(1-t)dt\right)^{\frac{1}{p}}\Bigg(|f''(b)|^q\int_0^1 h^s(t)dt\\&+m\left|f''\left(\frac{a+b}{2m}\right)\right|^q\int_0^1(1-h(t))^s dt\Bigg)^{\frac{1}{q}}\Bigg\}\end{aligned}\tag{10}$$

with L as before.

Proof. Using Hölder Inequality from I_1 we have

$$\begin{aligned}&\int_0^1 w(t)\left|f''\left(\frac{a+b}{2}t+(1-t)a\right)\right|dt\\&\le\left(\int_0^1 w^p(t)dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(\frac{a+b}{2}t+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\\&\le\left(\int_0^1 w^p(t)dt\right)^{\frac{1}{p}}\Bigg(\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 h^s(t)dt\\&\quad+m\left|f''\left(\frac{a}{m}\right)\right|^q\int_0^1(1-h(t))^s dt\Bigg)^{\frac{1}{q}}.\end{aligned}\tag{11}$$

Analogously, from I_2

$$\begin{aligned}&\int_0^1 w(1-t)\left|f''\left(bt+(1-t)\frac{a+b}{2}\right)\right|dt\\&\le\left(\int_0^1 w^p(1-t)dt\right)^{\frac{1}{p}}\Bigg(|f''(b)|^q\int_0^1 h^s(t)dt\\&\quad+m\left|f''\left(\frac{a+b}{2m}\right)\right|^q\int_0^1(1-h(t))^s dt\Bigg)^{\frac{1}{q}}.\end{aligned}\tag{12}$$

From (11) and (12) in Lemma 1 we arrive at the desired result. □

Remark 7. If $s=1$ and $w(t)=t^2$ we obtain the Theorem 2.4 of [40], and we say before, the Remark 2.5 of the cited paper still valid.

With $w(z)=\frac{z^{\alpha+1}}{\Gamma(\alpha)}$ we obtain a new result for Riemann-Liouville integrals:

Corollary 2. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° where $a,b\in I$ with $a<b$. If $f''\in L[a,b]$, then the*

followng equality holds:

$$\left|\frac{\alpha(1+\alpha)2^{\alpha-1}}{(b-a)^{\alpha}}\left(J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right)\right.$$
$$\left.-\frac{1+\alpha}{\Gamma(\alpha)}f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)^2}{8\Gamma(\alpha)}\left[\int_0^1\left[t^{\alpha+1}\left|f''\left(\frac{a+b}{2}\right)\right|\right.\right.$$
$$\left.+(1-t)^{\alpha+1}\left|f''(b)\right|\right]h^s(t)dt$$
$$+m\int_0^1\left[t^{\alpha+1}\left|f''\left(\frac{a}{m}\right)\right|+(1-t)^{\alpha+1}\left|f''\left(\frac{a+b}{2}\right)\right|\right]$$
$$\left.\cdot(1-h(t))^s dt\right].$$

Theorem 3. *Let $f : I \subset [0,d) \to \mathbb{R}$ b a differentiable mapping on I°, such that $f'' \in L[a,b]$, where $a, b \in I$ with $0 \leq a < b < \infty$ and w has second derivative integrable on I. If $|f''|^q$ is (h,m)-convex on $[a,b]$, for some fixed $m \in (0,1]$, $t \in [0,1]$ and $q \geq 1$, then the following inequality holds:*

$$|L| \leq \frac{(b-a)^2}{8}\left\{\left(\int_0^1 w(t)dt\right)^{1-\frac{1}{q}}\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q\right.\right.$$
$$\left.\times\int_0^1 w(t)h^s(t)dt+m\left|f''\left(\frac{a}{m}\right)\right|^q\int_0^1 w(t)(1-h(t))^s dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 w(1-t)dt\right)^{1-\frac{1}{q}}\left(\left|f''(b)\right|^q\int_0^1 w(1-t)h^s(t)dt\right.$$
$$\left.\left.+m\left|f''\left(\frac{a+b}{2m}\right)\right|^q\int_0^1 w(1-t)(1-h(t))^s dt\right)^{\frac{1}{q}}\right\} \tag{13}$$

with L as before.

Proof. By mean of power mean inequality, from I_1 we obtain

$$\int_0^1 w(t)\left|f''\left(\frac{a+b}{2}t+(1-t)a\right)\right|dt \leq \left(\int_0^1 w(t)dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 w(t)\left|f''\left(\frac{a+b}{2}t+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq\left(\int_0^1 w(t)dt\right)^{1-\frac{1}{q}}\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 w(t)h^s(t)dt\right.$$
$$\left.+m\left|f''\left(\frac{a}{m}\right)\right|^q\int_0^1 w(t)(1-h(t))^s dt\right)^{\frac{1}{q}}. \tag{14}$$

In the case of I_2 the use of this inequality gives us

$$\int_0^1 w(1-t)\left|f''\left(bt+(1-t)\frac{a+b}{2}\right)\right|dt$$
$$\leq\left(\int_0^1 w(1-t)dt\right)^{1-\frac{1}{q}}\left(\left|f''(b)\right|^q\int_0^1 w(1-t)h^s(t)dt\right.$$
$$\left.+m\left|f''\left(\frac{a+b}{2m}\right)\right|^q\int_0^1 w(1-t)(1-h(t))^s dt\right)^{\frac{1}{q}}. \tag{15}$$

From (14) and (15) we can arrive to the desired result. □

Remark 8. If $s = 1$ and $w(t) = t^2$ we obtain an extension of Theorem 2.6 of [40] and the Remark 2.7 of above paper is still valid.

Remark 9. For Riemann-Liouville integrals we have the following result, i.e., $w(t) = \frac{z^{\alpha+1}}{\Gamma(\alpha)}$:

$$\left|\frac{\alpha(1+\alpha)2^{\alpha-1}}{(b-a)^{\alpha}}\left(J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right)\right.$$
$$\left.-\frac{1+\alpha}{\Gamma(\alpha)}f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)^2}{8}\frac{1}{(2+\alpha)^{1-\frac{1}{q}}}\left[\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q\right.\right.$$
$$\times\int_0^1 t^{1+\alpha}h^s(t)dt+m\left|f''\left(\frac{a}{m}\right)\right|^q$$
$$\left.\times\int_0^1 t^{1+\alpha}(1-h(t))^s dt\right)^{\frac{1}{q}}$$
$$+\left(\left|f''(b)\right|^q\int_0^1 t^{1+\alpha}h^s(t)dt\right.$$
$$\left.\left.+m\left|f''\left(\frac{a+b}{2m}\right)\right|^q\int_0^1 t^{1+\alpha}(1-h(t))^s dt\right)^{\frac{1}{q}}\right].$$

Theorem 4. *Let $f : I \subset [0,d) \to \mathbb{R}$ b a differentiable mapping on I° such that $f'' \in L[a,b]$, where $a, b \in I$ with $0 \leq a < b < \infty$. If $|f''|^q$ is (h,m)-convex on $[a,b]$, for some fixed $m \in (0,1]$ and $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:*

$$|L| \leq \frac{(b-a)^2}{8}\left(\frac{\int_0^1 w^p(t)dt}{p}+\frac{1}{q}\left[\left|f''\left(\frac{a+b}{2}\right)\right|^q\right.\right.$$
$$\left.\times\int_0^1 h^s(t)dt+m\left|f''\left(\frac{a}{m}\right)\right|^q\int_0^1(1-h(t))^s dt\right]$$
$$+\frac{\int_0^1 w^p(1-t)dt}{p}+\frac{1}{q}\left[\left|f''(b)\right|^q\int_0^1 h^s(t)dt\right.$$
$$\left.\left.+m\left|f''\left(\frac{a+b}{2m}\right)\right|^q\int_0^1(1-h(t))^s dt\right]\right). \tag{16}$$

Proof. Taking into account the Young Inequality we have

$$\int_0^1 w(t)\left|f''\left(\frac{a+b}{2}t+(1-t)a\right)\right|dt$$
$$+\int_0^1 w(1-t)\left|f''\left(bt+(1-t)\frac{a+b}{2}\right)\right|dt$$
$$\leq\int_0^1\left[\frac{w^p(t)}{p}+\frac{\left|f''\left(\frac{a+b}{2}t+(1-t)a\right)\right|^q}{q}\right]dt$$
$$+\int_0^1\left[\frac{w^p(1-t)}{p}+\frac{\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q}{q}\right]dt$$

$$
\begin{aligned}
&= \int_0^1 \frac{w^p(t)}{p}dt + \frac{1}{q}\int_0^1 \left| f''\left(\frac{a+b}{2}t + (1-t)a\right)\right|^q dt \\
&+ \int_0^1 \frac{w^p(1-t)}{p}dt + \frac{1}{q}\int_0^1 \left| f''\left(tb + (1-t)\frac{a+b}{2}\right)\right|^q dt \\
&\le \int_0^1 \frac{w^p(t)}{p}dt + \frac{1}{q}\left[\left|f''\left(\frac{a+b}{2}\right)\right|^q \int_0^1 h^s(t)dt \right.\\
&\left. + m\left|f''\left(\frac{a}{m}\right)\right|^q \int_0^1 (1-h(t))^s dt\right] \\
&+ \int_0^1 \frac{w^p(1-t)}{p}dt + \frac{1}{q}\left[\left|f''(b)\right|^q \int_0^1 h^s(t)dt \right.\\
&\left. + m\left|f''\left(\frac{a+b}{2m}\right)\right|^q \int_0^1 (1-h(t))^s dt\right].
\end{aligned}
$$

Result we wanted to achieve. □

Remark 10. With $s = 1$ we obtain an extension of Theorem 2.8 of [40] and, as before, the Remark 2.9 of above paper is still valid.

Remark 11. If $w(t) = t^{1+\alpha}$, we obtain the following result valid for Riemann-Liouville fractional integrals:

$$
\begin{aligned}
&\left|\frac{\alpha(1+\alpha)2^{\alpha-1}}{(b-a)^\alpha}\left(J^\alpha_{a+}f\left(\frac{a+b}{2}\right) + J^\alpha_{b-}f\left(\frac{a+b}{2}\right)\right)\right. \\
&\left. - \frac{1+\alpha}{\Gamma(\alpha)}f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^2}{8} + \left\{\frac{1}{(\alpha(p+1)+1)}\right. \\
&\times \left[2 + \frac{1}{q}\left(\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q + \left|f''(b)\right|^q\right)\int_0^1 h^s(t)dt\right.\right. \\
&\left.\left.\left. + m\left(\left|f''\left(\frac{a}{m}\right)\right|^q + \left|f''\left(\frac{a+b}{2m}\right)\right|^q\right)\int_0^1 (1-h(t))^s dt\right)\right]\right\}.
\end{aligned}
$$

III. Conclusions

In this paper, we have obtained new versions of the well-known Hermite-Hadamard Inequality, via weighted integral. One of the features of the generality of our results is given use of a general weight in the integral operator used, which does not offer us a generalization, but rather "families" of inequalities, since the weight can lead us to classical Riemann integrals or to fractional integrals of the Riemann-Liouville type as we have pointed out above throughout the work. If we add to this the notion of convexity, which encompasses many known definitions, we have an idea of the breadth and scope of the results obtained.

References

[1] Valdés, J. E. N., Rabossi, F., & Samaniego, A. D. (2020). Convex functions: Ariadne's thread or Charlotte's Spiderweb?. *Advanced Mathematical Models & Applications, 5*(2), 176-191

[2] Hermite, C. (1883). Sur deux limites d'une intégrale définie. *Mathesis, 3*(82), 1.

[3] Hadamard, J. (1893). Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann. *Journal De Mathématiques Pures Et Appliquées, 9*, 171-215.

[4] Ali, M. A., Valdés, J. E. N., Kashuri, A., & Zhang, Z. (2020). Fractional non conformable Hermite-Hadamard inequalities for generalized φ-convex functions. *Fasciculi Mathematici, 64*, 5-16.

[5] Hassani, M., & Eghbali Amlashi, M. (2021). More on the Hermit-Hadamard inequality. *International Journal of Nonlinear Analysis and Applications, 12*(2), 2153-2159.

[6] Bayraktar, B., Butt, S. I., Napoles, J. E., & Rabossi, F. (2024). Some new estimates of integral inequalities and their applications. *Ukrains' kyi Matematychnyi Zhurnal, 76*(2), 159-178.

[7] B. Bayraktar, J. E. Nápoles V., A note on Hermite-Hadamard integral inequality for (h, m)-convex modified functions in a generalized framework, submited.

[8] Bermudo, S., Korus, P., & Nápoles Valdés, J. E. (2020). On q-Hermite–Hadamard inequalities for general convex functions. *Acta Mathematica Hungarica, 162*, 364-374.

[9] Butt, S. I., Bayraktar, B., & Valdes, J. E. N. (2023). Some new inequalities of Hermite-Hadamard type via Katugampola fractional integral. *Punjab University Journal of Mathematics, 55*(7-8), 269-289.

[10] El Farissi, A. (2010). Simple proof and refinement of Hermite-Hadamard inequality. *Journal of Mathematical Inequalities, 4*(3), 365-369.

[11] Juan Gabriel Galeano Delgado, Juan E. Nápoles Valdés, Edgardo Pérez Reyes, Some inequalities of the Hermite-Hadamard type for two kinds of convex functions, Revista Colombiana de Matemáticas, 57(2023), 43-55

[12] Guzman, P. M., Lugo, L. M., Nápoles Valdés, J. E., & Vivas-Cortez, M. (2020). On a new generalized integral operator and certain operating properties. *Axioms, 9*(2), 69.

[13] GUZM, P. M., VALD, J. E. N., & Gasimov, Y. S. (2021). *Integral Inequalities Within the Framework of Generalized Fractional Integrals,11*(1), 69-84.

[14] Guzmán, P. M., Valdés, J. E. N., & Stojiljkovic, V. (2023). New extensions of Hermite–Hadamard inequality. *Contrib. Math, 7*, 60-66.

[15] Kórus, P., Valdés, J. E. N., & Bayraktar, B. (2023). Weighted Hermite–Hadamard integral inequalities for general convex functions. *Mathematical Biosciences and Engineering, 20*, 19929-19940.

[16] Marinescu, D. Ş., & Monea, M. (2020). A very short proof of the Hermite–Hadamard inequalities. *The American Mathematical Monthly, 127*(9), 850-851.

[17] Nápoles, J., & Bayraktar, B. (2024). Generalized Fractional Integral Inequalities for (h, m, s)-Convex Modified Functions of Second Type. *Sahand Communications in Mathematical Analysis, 21*(2), 69-82.

[18] Nápoles Valdés, J. E., Rodríguez, J. M., & Sigarreta, J. M. (2019). New Hermite–Hadamard type inequalities involving non-conformable integral operators. *Symmetry, 11*(9), 1108.

[19] Simic, S. (2018). Some refinements of Hermite-Hadamard inequality and an open problem. *Kragujevac Journal of Mathematics, 42*(3), 349-356.

[20] Bruckner, A. M., & Ostrow, E. (1962). Some function classes related to the class of convex functions. *Pacific Journal of Mathematics, 12*, 1203-1215.

[21] Dragomir, S. S., Pecaric, J., & Persson, L. E. (1995). Some inequalities of Hadamard type. *Soochow Journal of Mathematics, 21*, 335-341.

[22] Toader, G. (1985). Some generalizations of the convexity. In *Proceedings of the Colloquium on Approximation and Optimization, University Cluj-Napoca* (pp. 329-338).

[23] Breckner, W. W. (1978). Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen. *Publications de l'Institut Mathématique, 23*, 13-20.

[24] Hudzik, H., & Maligranda, L. (1994). Some remarks on s-convex functions. *Aequationes Mathematicae, 48*(1), 100-111.

[25] Park, J. (2011). Generalization of Ostrowski-type inequalities for differentiable real (s, m)-convex mappings. *Far East Journal of Mathematical Sciences, 49*, 157-171.

[26] Bilal, M., Imtiaz, M., Khan, A. R., Khan, I. U., & Zafran, M. Generalized Hermite-Hadamard inequalities for s-convex functions in the mixed kind. Submitted.

[27] Mihesan, V. G. (1993). A generalization of the convexity. In *Seminar on Functional Equations, Approximation and Convexity*, Cluj-Napoca, Romania.

[28] Xi, B. Y., Gao, D. D., & Qi, F. (2020). Integral inequalities of Hermite-Hadamard type for (α, s)-convex and (α, s, m)-convex functions. *Italian Journal of Pure and Applied Mathematics, 44*, 499-510.

[29] Özdemir, M. E., Akdemri, A. O., & Set, E. (2016). On (h, m)-convexity and Hadamard-type inequalities. *Transylvanian Journal of Mathematics and Mechanics, 8*(1), 51-58.

[30] Bayraktar, B., & Nápoles Valdés, J. E. (2022). New generalized integral inequalities via (h, m)-convex modified functions. *Izvestiya Instituta Matematiki i Informatiki, 60*, 3-15.

[31] Bayraktar, B., & Nápoles Valdés, J. E. (2022). Integral inequalities for mappings whose derivatives are (h, m, s)-convex modified of second type via Katugampola integrals. *University of Craiova Series: Mathematics and Informatics, 49*, 371-383.

[32] Mubeen, S., & Habibullah, G. M. (2012). k-fractional integrals and applications. *International Journal of Contemporary Mathematical Sciences, 7*, 89-94.

[33] Jarad, F., Abdeljawad, T., & Shah, T. (2020). On the weighted fractional operators of a function with respect to another function. *Fractals, 28*(8), 2040011.

[34] Sarikaya, M. Z., & Ertugral, F. (2020). On the generalized Hermite-Hadamard inequalities. *Annals of the University of Craiova, Mathematics and Computer Science Series, 47*(1), 193-213.

[35] Jarad, F., Ugurlu, E., Abdeljawad, T., & Baleanu, D. (2017). On a new class of fractional operators. *Advances in Difference Equations, 2017*, 247.

[36] Khan, T. U., & Khan, M. A. (2019). Generalized conformable fractional integral operators. *Journal of Computational and Applied Mathematics, 346*, 378-389.

[37] Mehmood, S., Nápoles Valdés, J. E., Fatima, N., & Shahid, B. (2021). Some new inequalities using conformable fractional integral of order β. *Journal of Mathematical Extension, 15*(SI-NTFCA), 33-1-22.

[38] Tomar, M., Set, E., & Sarikaya, M. Z. (2016). Hermite-Hadamard type Riemann-Liouville fractional integral inequalities for convex functions. *AIP Conference Proceedings, 1726*, 020035.

[39] Özdemir, M. E., Yıldız, C., Akdemri, A. O., & Set, E. (2013). On some inequalities for s-convex functions and applications. *Journal of Inequalities and Applications, 2013*, 333.

[40] Cakaloglu, M. N., Aslan, S., & Akdemir, A. O. (2021). Hadamard type integral inequalities for differentiable (h, m)-convex functions. *Eastern Anatolian Journal of Science, 7*(1), 12-18.

•••